Strings, quantum gravity and non-commutative geometry on the lattice

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I review recent progress in understanding non-perturbative aspects of string theory, quantum gravity and non-commutative geometry using lattice methods.

1. Introduction

Lattice methods have been very successful dealing with string theory, quantum gravity and non-commutative field theory in the following sense: whenever the bosonic theory is known to exist, the lattice formulation has
(1) provided a rigorous definition of the theory,
(2) a scaling limit where the continuum theory can be defined and which can be analyzed by numerical and/or analytical methods,
(3) often been superior to a continuum approach even from an analytical point of view.

Examples are 2d quantum gravity and non-critical strings with a central charge $c \leq 1$, non-commutative Yang-Mills theory (which was in fact discovered on the lattice as early as 1983 [1,2]), and 3d quantum gravity formulated as a Turaev-Viro state-sum. There are also examples where lattice attempts to formulate a non-perturbative theory have failed. But with the advantage of hindsight we now understand that the message from the lattice was correct even in these cases. One such example is the bosonic string theory in dimensions $d > 1$ [3–6]. Probably there exists no non-tachyonic bosonic string theory in dimensions $d > 1$.

The examples mentioned are non-trivial in the sense that they deal with diffeomorphism-invariant theories and one would not expect the lattice to be the best way to regularize such theories. In fact, the situation seems to be the opposite: the lattice framework introduced by the use of "dynamical triangulations" seems to work very well for the above-mentioned bosonic theories.

However, there are two fundamental areas where the lattice approach cannot yet claim success: four-dimensional quantum gravity and superstring theory. In the first case we do not know if there exists a consistent non-perturbative theory of quantum gravity, formulated entirely in terms of the gravitational fields, or if the theory has to be embedded in a larger theory like superstring theory. However, *if* a purely bosonic theory of four-dimensional quantum gravity exists, it follows from the remarks above that the lattice framework should be ideal for a non-perturbative definition. I will discuss a new lattice approach, called *Lorentzian simplical quantum gravity* in Sec. 4. In the case of superstring theories, lattice formulations have been stalled by the inability to implement supersymmetry in the correct way on the lattice. While a number of attempts to implement worldsheet supersymmetry on random lattices have failed, it has also been known that the chances were better if one used a Green-Schwarz formalism. In that case the supersymmetry is a *space-time* supersymmetry rather than a world sheet supersymmetry and thus not in direct conflict with an underlying worldsheet lattice. The obstacle came in that case from the so-called κ-symmetry which has not yet been implemented as an exact symmetry in the lattice approach, and it is unclear if it will automatically be satisfied

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when the lattice spacing is taken to zero.

The so-called "second string revolution" which pointed to unexpected and sometimes non-perturbative connections between the five different superstring theories and the so-called M-theory, led to the quest for a *non-perturbative* definition of these theories. The *new* matrix model approach to M-theory and type IIB superstrings initiated by Banks, Fischler, Shenker and Susskind (the BFSS matrix model [7]), and by Ishibashi, Kawai, Kitazawa and Tsuchiya (the IKKT matrix model, [8]), respectively, were attempts to provide such a definition. In these discretized models supersymmetry was implemented in a way which avoided the problem with κ-symmetry. The "continuuum limits" of the models correspond to N, the size of the matrices, going to infinity. However, even for finite N the concept of supersymmetry has a well defined meaning in these models. In this respect they differ from previous attempts to introduce supersymmetry and are more promising.

2. Superstrings on the lattice

2.1. Theory

In the so-called Schild gauge the action of a type IIB superstring can be written as the following expression (in dimensions $D=4,6$ and 10):

$$S=\int d^2\xi\sqrt{g}\Big(\frac{1}{4}\{X^\mu,X^\nu\}^2-\frac{i}{2}\bar\psi\Gamma_\mu\{X^\mu,\psi\}\Big), \quad (1)$$

where $\{X,Y\}$ denotes the Poisson bracket between the variables $X(\xi_1,\xi_2)$ and $Y(\xi_1,\xi_2)$ and Γ_μ are suitably defined γ-matrices. The idea behind the IKKT matrix model is to replace worldsheet variables by $N\times N$ matrices:

$$X^\mu(\xi_1,\xi_2)\to X^\mu_{ij}, \quad \psi^\alpha(\xi_1,\xi_2)\to\psi^\alpha_{ij}, \qquad (2)$$

In this way the worldsheet coordinates (ξ_1,ξ_2) are mapped into the matrix indices (i,j) and the worldsheet is replaced by the "matrix lattice" (i,j). The continuum limit should be obtained in the limit $N\to\infty$. As usual when going from a classical theory to a quantum theory we replace the Poisson brackets by commutators:

$$\{X,Y\}\to -i[X,Y], \qquad (3)$$

and in this way the IKKT action for the type IIB superstring becomes:

$$S=-\mathrm{Tr}\,\Big(\frac{1}{4}[X^\mu,X^\nu]^2+\frac{1}{2}\bar\psi\Gamma_\mu[X^\mu,\psi]\Big). \qquad (4)$$

The corresponding partition function is

$$Z_N=\int d\psi d\bar\psi dX_\mu\, e^{-S[\bar\psi,\psi,X_\mu]}. \qquad (5)$$

As described by IKKT, this model has even for finite N an invariance which might be called supersymmetry. The (bosonic part of the) classical equations of motion is

$$[X_\mu,[X_\mu,X_\nu]]=0, \qquad (6)$$

and the simplest solution is

$$[X^{cl}_\mu,X^{cl}_\nu]=0, \quad (\psi=0). \qquad (7)$$

This means that the matrices X^{cl}_μ can be simultaneously diagonalized and the *spray* of eigenvalues in R^{10} viewed as the points defining classical space-time.

The valleys $[X_\mu,X_\nu]=0$ make the existence of the integral (5) non-trivial. The convergence of the integral for all N in dimensions $D=4,6,10$ (these are the dimensions $D>3$ where classical supersymmetry of the Green-Schwarz superstring can be formulated) was established in [9].

In a non-perturbative closed string theory *the dimensionality of real space-time should be determined dynamically*. We would like to see that the typical "quantum" matrices X_μ dominating the matrix integral (5) are approximately diagonalizable and that the (suitably defined)*spray of eigenvalues* in the large-N limit constitutes a four-dimensional manifold: our world.

In order to make this more quantitative one can define the "space-time" uncertainty Δ as a measure of the ten matrices not being simultaneously diagonalizable:

$$\Delta^2=\frac{1}{N}\Big(\mathrm{Tr}\;X^2_\mu-\max_{U\in SU(N)}\sum_i(UX_\mu U^\dagger)^2_{ii}\Big). \qquad (8)$$

$\Delta^2=0$ if the X_μ's are simultaneously diagonalizable. Let us take the $U\equiv U_{max}$ which minimizes the RHS of (8). We define the "space-time coordinates"

$$(x_\mu)_i\equiv\Big(U_{max}X_\mu U^\dagger_{max}\Big)_{ii}. \qquad (9)$$

With this definition we can now define the *extension of space-time R*:

$$R^2 = \frac{1}{N}\langle \text{Tr } X_\mu^2 \rangle = \frac{\left\langle \sum_{i<j}(x_i^\mu - x_j^\mu)^2 \right\rangle}{N(N-1)/2}, \qquad (10)$$

as well as the density of the spray of space-time points

$$\rho(r) = \frac{\left\langle \sum_{i<j} \delta(r - \sqrt{(x_i^\mu - x_j^\mu)^2} \right\rangle}{N(N-1)/2}. \qquad (11)$$

When $N=2$ we have an $SU(2)$ matrix model and it has been proven that [10]

$$\rho(r) \sim r^{-2D+5}, \quad r \text{ large.} \qquad (12)$$

There exists good arguments in favor of (12) for all values of N [10,11,9].

The simplest way to probe the effective dimension of the space-time dynamically generated from the distribution of x_i^μ's is to look at the "moments of inertia" for such a space-time:

$$T^{\mu\nu} = \frac{2}{N(N-1)} \sum_{i<j} \left\langle (x_i^\mu - x_j^\mu)(x_i^\nu - x_j^\nu) \right\rangle. \qquad (13)$$

This is a 10×10 matrix and the principal moments of inertia for the distribution of x_i^μ are the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{10}$.

Our four-dimensional world should appear as a flat four-dimensional pancake distribution of x_i^μ in the large N limit, i.e. the four first eigenvalues of $T_{\mu\nu}$ should separate from the rest such that one has a spontaneous breakdown of the ten-dimensional Lorentz invariance in the large-N limit. Further, one would like the ratio $\Delta/R \to 0$ in the large N limit, such that it makes sense to talk about a classical background.

The power behavior (12) of $\rho(r)$ is a source of ambiguity in defining R and $T_{\mu\nu}$, since $\langle \text{Tr } X^{2n} \rangle = \infty$ for sufficiently large n. This is in contrast to the situation where $\rho(r)$ falls off exponentially.

2.2. Numerical results

Even if the non-perturbative regularization of the type IIB superstring has resulted in a finite-dimensional "lattice"-theory, the lattice being the entries of the $N\times N$ matrices X_μ, the theory is not well suited for numerical simulations since it has fermions. One can integrate out the fermions from (5) and obtain

$$Z_N = \int \prod_\mu \mathrm{d}X_\mu \; e^{\frac{1}{4}\text{Tr } [X_\mu, X_\nu]^2} \; \det M(X), \qquad (14)$$

where $M(X)$ is a $(N^2-1)\times(N^2-1)$ matrix. For generic X it is complex if the dimension D of space-time is 6 or 10, and real and positive if $D=4$. The first question we want to address is whether we observe any trace of spontaneous breakdown of Lorentz invariance. Analytical arguments in favor of a dimensional reduction to four dimensions have been given [8], using a one-loop approximation, but although very encouraging they need to be substantiated. Computer simulations, using the same one-loop approximation to the action, can be performed if one drops the phase of the determinant in (14). Let us define

$$\det M_\nu(X) = |\det M(X)| \; e^{\nu\Gamma(X)}. \qquad (15)$$

$\nu=1$ corresponds to the physical situation we want to solve. $\nu=0$ corresponds to the situation we can simulate on the computer. There exist arguments in favor of a dimensional reduction if $\nu=\infty$ [12]. Consequently, if we observe symmetry breaking for $\nu=0$ it would be a strong argument in favor of symmetry breaking for $\nu=1$. In Fig. 1 we show the result of the measurement of the eigenvalues of $T_{\mu\nu}$ in the case of $\nu=0$. For details of the computer simulations shown here, see [13]. From the results shown in Fig. 1 there is no trace of a spontaneous symmetry breakdown. This is an indication that the phase of the determinant may play a decisive role in a possible symmetry breakdown. However, in [11] the model was investigated in $D=4$ where the fermionic determinant is real and positive and a dimensional reduction to *one dimension* was observed. This result highlights the ambiguity associated with power law distributions like (12). Indeed, $T_{\mu\nu}$ as defined in (13) is divergent if $D=4$ and one of the eigenvalues of $T_{\mu\nu}$ diverges. According to [11] this is a reflection of the fact that the *tail* of the distribution $\rho(r)$ is caused by aligned one-dimensional configurations of x_i^μ's. It depends on the choice of

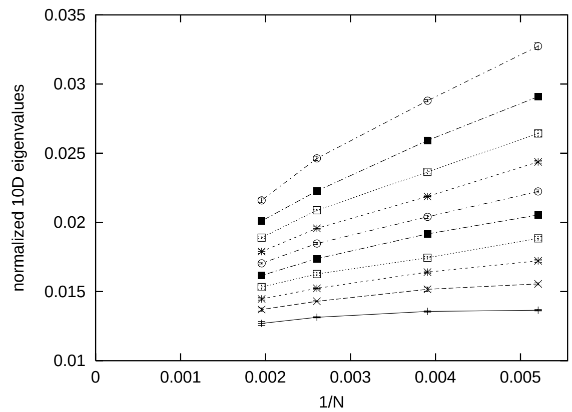

Figure 1. The 10 eigenvalues of $T_{\mu\nu}$, extracted from Monte Carlo simulations of the one-loop approximation to (14) with $M(X)$ replaced by $|M(X)|$, and plotted as a function of $1/N$.

observables how sensitive they are to this tail. As an example one can define a modified, converging $T^{new}_{\mu\nu}$ in $D=4$:

$$T^{new}_{\mu\nu} = \frac{2}{N(N-1)} \sum_{i<j} \left\langle \frac{(x_i^\mu - x_j^\mu)(x_i^\nu - x_j^\nu)}{\sqrt{(x_i - x_j)^2}} \right\rangle . \quad (16)$$

Using the definition (16) one does not see any trace of spontaneously symmetry breaking in the case of $D=4$. This is illustrated in Fig. 2. Also

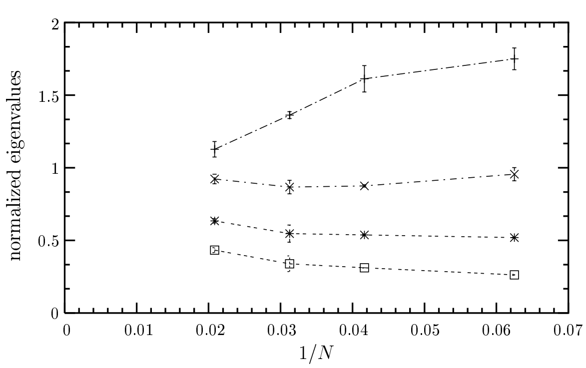

Figure 2. The eigenvalues of $T^{new}_{\mu\nu}$ in the case $D=4$, plotted as a function of $1/N$

measurements of Wilson loops, which have the interpretation as expectation values of fundamental strings in the theory, do not show any sign of spontaneous symmetry breaking [13].

This leaves the question concerning the spontaneous breaking of Lorentz invariance in the model unsettled. More work is needed in order to understand whether or not the signal of breaking depends on the observables chosen, and if it *does*, how to interpret this. In addition it would of course be very desirable if one could perform simulations in $D=10$, using the full complex determinant (or more precisely, Pfaffian). It would require handling a complex action, a notorious problem in computer simulations. Progress in dealing with a problem of the kind encountered here has been been made recently [14].

2.3. The BFSS matrix model

The BFSS matrix model was conjectured to provide a non-perturbative definition of M-theory. Thus it is an eleven-dimensional theory where the matrices $X(t)$ depend on time. The problem with fermions also exists in this model, and it is more difficult to simulate than the IKKT model, since it is a matrix chain model, not a single matrix model. A first attempt has been reported at this conference [15]. Finally, extensive numerical studies of a mean field approximation to the BFSS model have been performed recently [16].

3. Non-commutative gauge theories on the lattice

Non-commutative Yang-Mills theory was discovered on the lattice [1,2]. In the context of the type IIB superstrings discussed in the last section they occur by expanding around a classical solution to (6) different from (7) [17,18], namely one satisfying

$$[X_\mu^{cl}, X_\nu^{cl}] = iC_{\mu\nu}, \quad (17)$$

where $C_{\mu\nu}$ are c-numbers. Note that (17) requires the matrices X_μ^{cl} to be infinite-dimensional. Thus one has to work directly at the $N=\infty$ limit. Non-commutative field theories and in particular non-commutative gauge theories have recently been studied intensively because they appear both in type IIB string theories and in open string theory (where the gauge theory, living on the D-branes where the open strings end, becomes non-commutative in the presence of an external so-called $B_{\mu\nu}$ field). But non-commutative gauge theories can be defined and studied independently of string theory and were indeed studied before they appeared in string theory.

As already noted when the theories were discovered, a natural regularization of non-commutative YM theory is provided by the TEK lattice model [1] by changing from matrices X_μ to the exponentials $U_\mu = \mathrm{e}^{iaX_\mu}$. This has the following virtues: (1) It respects non-commutative gauge symmetry in the same way as ordinary gauge symmetry is respected on the lattice. (2) It can be formulated for finite N (this is what provides the regularization of the theory) and (3) it preserves Morita equivalence.

The TEK model is $U(N)$ gauge theory on a hypercube with twisted boundary conditions, given by the partition function

$$Z_{TEK}(U(N)) = \int \prod_\mu \mathrm{d}U_\mu \; \mathrm{e}^{\frac{1}{2g^2}\sum_{\mu<\nu}\left(Z_{\mu\nu}\,\mathrm{Tr}\; U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + \mathrm{H.C.}\right)}. \tag{18}$$

where $Z_{\nu\mu} = \mathrm{e}^{4\pi n_{\mu\nu}/N} \in Z_N$. The classical vacuum of the model becomes non-trivial and can be expressed in terms of the so-called twist eaters Γ_μ:

$$U^{cl}_\mu = \Gamma_\mu, \quad \Gamma_\mu \Gamma_\nu = Z_{\mu\nu} \Gamma_\nu \Gamma_\mu. \tag{19}$$

This equation replaces (17). Let $N = L^{D/2}$, D even. Then the twist eaters can be used to construct the so-called Weyl map from the matrices U^{ij}_μ to functions $\mathcal{U}_\mu(x)$ with arguments x on the hypercubic lattice Z^D_L. This is the inverse map to (2). By this map matrix multiplication is mapped to the star product of functions:

$$U^{ij} \to \mathcal{U}(x), \quad \sum_k U^{ik}_1 U^{kj}_2 \to \mathcal{U}_1(x) * \mathcal{U}_2(x), \tag{20}$$

where the star product $*$ is defined as

$$A(x) * B(x) = \sum_{y,z} \mathrm{e}^{2i\theta^{-1}_{\mu\nu} y_\mu z_\nu} A(x+y) B(x+z), \tag{21}$$

and $\theta_{\mu\nu} \sim n_{\mu\nu}$ (see [19] for details). The partition function $Z_{TEK}(U(N))$ is mapped to

$$Z^{(NC)}_{L^D}(U(1)) = \int \prod_\mu \mathrm{d}\mathcal{U}_\mu(x)\; \mathrm{e}^{-S^{(NC)}(\mathcal{U})}, \tag{22}$$

$$S^{(NC)}(\mathcal{U}) = \frac{1}{2e^2} \sum_{\mu<\nu,x} \mathcal{U}_\mu(x) * \mathcal{U}_\nu(x+\hat{\mu}) * \mathcal{U}^\dagger_\mu(x+\hat{\nu}) * \mathcal{U}^\dagger_\nu(x), \tag{23}$$

where $e^2 = g^2 N$. This relation:

$$Z_{TEK}(U(N)) = Z^{(NC)}_{L^D}(U(1)), \tag{24}$$

which states that a $U(N)$ commutative gauge theory on a hypercube (a 1^D lattice) is equivalent to a non-commutative $U(1)$ theory on a L^D lattice, $L^D = N$, is the simplest example of the exact Morita equivalence for lattice gauge theories [19]. Note that the number of degrees of freedom is the same: $D\,N^2$ for $U(N)$ on the hypercube and $D\,L^D$ for $\mathcal{U}$ for the non-commutative $U(1)$ theory on the L^D lattice. It illustrates that lattice gauge theory provides us with a natural framework for defining non-commutative field theories in a *non-perturbative* way.

4. Quantum gravity on the lattice

4.1. General considerations

As mentioned in the Introduction the lattice formalism of dynamical triangulations seems the ideal framework to address higher dimensional quantum gravity. Until now we have not been successful[2]. However, it might be that the problems encountered so far are more related to *Euclidean* gravity than to *quantum* gravity.

In two-dimensional quantum gravity the simplest direct approach to the theory of fluctuating geometries has been very successful. By rotating to Euclidean signature and regulating the sum over geometries by introducing the reparameterization-invariant lattice cut-off called dynamical triangulations, one was able to calculate generalized Hartle-Hawking wave functionals and correlation functions depending on the

[2] see however the contribution by Shinichi Horata at this conference for interesting progress.

geodesic distance [20]. However, simple generalizations to higher dimensions seem not to work. While four-dimensional quantum gravity may not exist without being embedded in a larger theory, this is not true for three-dimensional quantum gravity. This led to the suggestion [21], following an old idea by Teitelboim, that Euclidean quantum gravity might not be related to "real" Lorentzian quantum gravity, and that one should only include causal geometries in the sum over histories. The geometries which appear in the regularized version of such a theory were called *Lorentzian dynamical triangulations*, and each of these geometries has a well defined rotation to an Euclidean geometry. The opposite is not true: there are many Euclidean geometries which cannot be rotated to a Lorentzian geometry with a global causal structure. However, it implies that one *can* perform the summation over this restricted class of geometries in the "Euclidean sector", and rotate back after the summation has been done. This is the way we will treat the summation over histories in the following.

In two dimensions one can perform the summation over the class of Lorentzian geometries explicitly and obtain a theory which *differs* from Euclidean two-dimensional quantum gravity. The difference is best illustrated by considering what is called the proper-time propagator, where one sums over all geometries with the topology $S^1 \times [0,1]$, where the two spatial boundaries are separated by a proper time T. In the Lorentzian theory the spatial slices at a time $T' < T$ are characterized by the fact that the spatial topology always remains a circle. In two-dimensional Euclidean quantum gravity similar spatial slices corresponding to constant proper time split up into many (in the continuum limit infinitely many) disconnected "baby" universes, *each* having the topology S^1 [22].

In addition three-dimensional quantum gravity is interesting for the following reason: locally, the classical solution is just flat space, or in the case of a positive cosmological constant, 3d de Sitter space. If one expands around such a classical solution in order to quantize the theory one finds it is non-renormalizable. Nevertheless we know the theory has no dynamical *field degrees of freedom*, but only a finite number of degrees of freedom. Thus it can be quantized following different procedures, e.g. reduced phase space quantization. However, it remains unclear if anything is "wrong" in an approach where one performs the summation over fluctuating three-geometries and how such an approach deals with the seeming "non-renormalizability" of the theory of fluctuating geometries.

Like in two dimensions, also in three dimensions there will be a drastic difference between what we call Euclidean quantum gravity and Lorentzian quantum gravity. Loosely speaking, because of the restricted class of geometries which enters into the sum over histories in the case of Lorentzian quantum gravity, the quantum theory will be better behaved, and, contrary to the situation in a regularized Euclidean quantum gravity theory, one can define a continuum limit. Seemingly, Euclidean quantum gravity theory, as defined by dynamical triangulations, does not treat the conformal factor correctly (see [23] for a discussion.). It is a little surprising since the main success of the formalism in two dimensions precisely was the correct treatment of the conformal mode. The explanation may be related to the fact that the conformal mode in higher dimensions is also the cause of the *unboundedness* of the Euclidean Einstein action. In the Lorentzian theory one still has a conformal mode, but the geometries associated with this mode are less "singular" than the geometries one meets in the Euclidean theory (see [24] for details).

4.2. 3d Lorentzian dynamical triangulations

As mentioned above the proper-time propagator is a convenient object to study. We choose the simplest possible topology of space-time, $S^2 \times [0,1]$, so that the spatial slices of constant proper time have the topology of a two-sphere. Each spatial slice has an induced two-dimensional Euclidean geometry. In the formalism of Lorentzian dynamical triangulations the space of Euclidean 2d geometries is approximated by the space of 2d dynamical triangulations. This approximation is known to work well and in the limit where the lattice spacing (the length of the lattice links) goes to zero the continuum limit, i.e. quantum Liouville theory, is recovered. In order to obtain a three-dimensional triangulation of space-time we have to fill in the space-time between two successive spatial slices. This is done as follows: above (and below) each triangle at proper time $t = n\,a$, n an integer, we erect a tetrahedron with its tip at $t+a$, a so-called (3,1)-tetrahedron (if the tip is at $t-a$ a (1,3)-tetrahedron). Two tetrahedra which share a spatial link in the constant-t plane might be glued together along a common time-like triangle. Remaining free time-like triangles with either the spatial link in the constant-time slice at t and the tip at $t+a$, or the spatial link at the constant-time slice at $t+a$ and the tip at t, are glued together by so-called (2,2)-tetrahedra. They have a spatial link both in the constant-t slice and the constant $t+a$ slice. (2,2)-tetrahedra can also be glued to each other in all possible ways, the only restriction being that if we cut the triangulation in a spatial plane between t and $t+a$, the corresponding graph, which consists of triangles and squares (coming from cutting the (2,2)-simplexes) form a graph with spherical topology.

Summing over all such piecewise linear geometries with the Boltzmann weight given by the Einstein-Hilbert action defines the sum over geometries (see [24] for details). The partition function becomes (up to boundary terms)

$$Z(k_0, k_3, T) = \sum_T \mathrm{e}^{k_0 N_0(T) - k_3 N_3(T)}, \qquad (25)$$

where the summation is over the class of triangulations mentioned, $N_0(T)$ denotes the total number of vertices and $N_3(T)$ the total number of tetrahedra in the triangulation T. k_0 is inversely proportional to the bare inverse gravitational coupling constant, while k_3 is linearly related to the cosmological coupling constant.

4.3. Numerical simulations

The model (25) can be studied by Monte Carlo simulations (see [24] for details). There is only one phase[3]. Let us fix the total three-volume of space-time to be N_3, and let us take the total proper time T large. One observes the appearance of an "semiclassical" lump of universe, as shown in Fig. 3. As we change the bare gravitational coupling constant, the time-extent of the semiclassical lump decreases. However, all correlation functions we have studied can be mapped into each other by a simple rescaling of time and space directions. Further, we observe that a typical spatial volume $N_2(t)$ in the lump and the time-extent ΔT of the lump scale as:

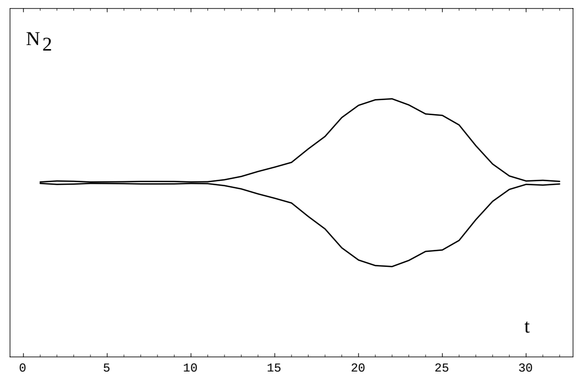

Figure 3. A snapshot of a three-dimensional configuration of space-time. The vertical axis is the spatial volume N_2, the horizontal axis the proper time t.

$$N_2 \sim N_3^{2/3}, \qquad \Delta T \sim N_3^{1/3}. \qquad (26)$$

This justifies the use of the word "semiclassical" in the description of the lump. In the computer simulations we observe that the center of mass of the lump moves around randomly. In addition there are fluctuations in the spatial volume. We have studied the fluctuations of *successive* spatial volume. The distribution of such spatial volumes is very well described by the formula

$$P(N_2(t), N_s(t+a)) \sim \mathrm{e}^{-c(k_0)\frac{(N_2(t+a)-N_2(t))^2}{N_2(t+a)+N_2(t)}}. \qquad (27)$$

The constant $c(k_0)$ decreases as k_0 increases (i.e. the bare gravitational coupling constant decreases). At the same time one can observe that

[3]In some previous studies we observed a phase transition for large k_0. This was caused by restrictions on the gluing of (2,2)-simplices. We have now dropped these restrictions.

the total number of (2,2)-simplices decreases, indicating that the (2,2)-simplices act as glue between successive spatial volumes.

Thus the leading terms in the effective action for the spatial volume of the model are given by

$$S_{eff}(N_2) = \int dt \,\Big(\frac{\dot{N}_2^2(t)}{N_2(t)} + \Lambda N_2(t)\Big). \tag{28}$$

This is exactly the classical Lorentzian action for the spatial volume in proper-time gauge, thereby supporting the semiclassical interpretation of the lump.

4.4. 3d Lorentzian gravity as a matrix model

If we slice our three-dimensional configurations, not at proper time t, but at proper time $t+a/2$ we will, as mentioned earlier, obtain a spherical graph with two types of triangles, coming from the spatial intersections of (1,3)- and (3,1)-tetrahedra, respectively. In addition the graph will contain squares coming from the (2,2)-tetrahedra. This class of graphs can be described by a two-matrix model:

$$Z = \int \mathrm{d}A\mathrm{d}B\, \mathrm{e}^{-N\mathrm{Tr}\, \left((A^2+B^2)-\alpha(A^3+B^3)-\beta ABAB\right)}. \tag{29}$$

A and B are $N\times N$ Hermitian matrices and the spherical graphs are selected in the large-N limit. The coupling constants α, β can be related to the gravitational coupling constants. While this matrix model has not been solved, there exists another, closely related matrix model where

$$A^3 + B^3 \to A^4 + B^4, \tag{30}$$

which *has* been solved [25]. This model has a simple interpretation in terms of "triangulations": the (3,1)- and (1,3)-tetrahedra are replaced by (4,1)- and (1,4)-pyramids in an obvious notation. This model, where the building blocks are pyramids and (2,2)-tetrahedra, is an equally good regularization of three-dimensional gravity. Again one can work out the relation between the coupling constants α, β and k_0, k_3 (see [26] for details). The matrix model is defined for sufficiently small values of α, β and has a critical line in the (β, α)-plane where the continuum limit is obtained, see Fig. 4. The three dotted curves approaching the critical line correspond to three different fixed values of the bare gravitational constant. The bare cosmological constant k_3 changes along the dotted lines to achieve its critical value $k_3^c(k_0)$ (which depend on k_0) at the critical line shown in the figure. Fine tuning of k_3 to $k_3^c(k_0)$ corresponds to a renormalization of the cosmological constant and the approach to the infinite volume limit of the theory: for $k_3 \to k_3^c(k_0)$ the expectation value of the number of tetrahedra or pyramids diverges. On the other hand there seems to be no need for a renormalization of k_0: it simply defines an overall scale for the model.

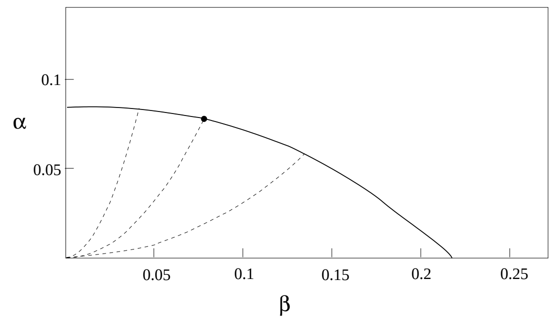

Figure 4. The phase diagram of 3d Lorentzian quantum gravity from matrix models

If we follow the critical line of the matrix model, starting at small values of β, it corresponds initially to a weak-coupling phase of three-dimensional quantum gravity where the bare gravitational coupling constant is small (k_0 large). This is the phase we have observed in the computer simulation. However, the matrix model undergoes a phase transition at $\alpha = \beta$, shown by a dot on Fig. 4, separating the weak-coupling phase from a strong-coupling phase, corresponding to large values of the gravitational coupling constant. In this phase the triangulations corresponding to the spatial slices at t and $t+a$ of topology S^2 disintegrate into many (in the continuum limit where $a \to 0$ into infinitely many) baby universes each of topology S^2, connected by a web of thin wormholes. This is possible because the matrix model admits more general configurations than were allowed in the computer simulations. The only requirement of the matrix model is that the combined graph at $t+a/2$ is spherical. The component coming from the spatial slice at t can actually be disconnected (and similar for the component coming from the spatial slice at $t+a$). It becomes a dynamical question what happens if one allows for such fluctuations. In the weakly coupled phase the spatial topology stays unchanged as S^2, but increasing the gravitational coupling constant space starts to be torn apart. When the gravitational coupling constant is sufficiently large a phase transition takes place and space disintegrates into baby universes, only connected by thin wormholes (see [26] for details). In this way the model provides a concrete realization of the ideas of Wheeler and Hawking of a quantum foam at short distances.

Defined in a non-perturbative way on the lattice, three-dimensional quantum gravity reveals a rich structure, and, as in two dimensions, the model can be analyzed by a fruitful interplay between numerical and analytic methods.

Hopefully, the same will be true in the case of four-dimensional Lorentzian quantum gravity defined via dynamical triangulations.

REFERENCES

1. A. Gonzalez-Arroyo and M. Okawa, Phys. Rev. D27 2397 (1983).
2. A. Gonzalez-Arroyo and C.P. Korthals Altes, Phys. Lett. B131 396 (1983).
3. V. A. Kazakov, A. A. Migdal and I. K. Kostov, Phys. Lett. B 157, 295 (1985).
4. F. David, Nucl. Phys. B 257, 543 (1985).
5. J. Ambjorn, B. Durhuus and J. Frohlich, Nucl. Phys. B 257, 433 (1985); Nucl. Phys. B 275 161 (1986).
6. J. Ambjorn, B. Durhuus, J. Frohlich and P. Orland, Nucl. Phys. B 270, 457 (1986).
7. T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. D55 5112 (1997).
8. N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B498 467 (1997). H. Aoki, A. Iso, H. Kawai, Y. Kitazawa, and A. Tsuchiya, Prog. Theor. Phys. 99 713 (1998).
9. P. Austing and J. Wheater, JHEP 0104 019 (2001).
10. W. Krauth, M. Staudacher et al., Phys. Lett. B453 253 (1999); Phys. Lett. B435 350 (1998); Phys. Lett. B431 31 (1998).
11. P. Bialas, Z. Burda, B. Petersson, J. Tabaczek, Nucl. Phys. B592 391 (2001); Nucl. Phys. B602 399 (2001).
12. J. Nishimura and G. Vernizzi, Phys. Rev. Lett. 85 4664 (2000).
13. J. Ambjørn, K.N. Anagnostopoulos, W. Bietenholz, J. Nishimura et al., JHEP 0007 013 (2000); JHEP 0007 011 (2000); hep-th/0104260.
14. K.N. Anagnostopoulos and J. Nishimura, hep-th/0108041.
15. P. Bialas, R.A. Janik and J. Wosiek, hep-lat/0105031; Acta Phys. Polon. B32 2143 (2001).
16. D. Kabat, G. Lifschytz et al. Nucl. Phys. B571 419 (2000); hep-th/0105171.
17. A. Connes, M. Douglas and A. Schwarz, JHEP 9802 003 (1998).
18. H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, Nucl. Phys. B565 176 (2000).
19. J. Ambjørn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, JHEP 0005 023 (2000); Phys. Lett. B480 399 (2000); JHEP 9911 029 (1999).
20. J. Ambjørn, J. Jurkiewicz and Y. Makeenko, Phys. Lett. B251 517 (1990).
21. J. Ambjørn and R. Loll, Nucl. Phys. B536 407 (1998).
22. H. Kawai, N. Kawamoto, T. Mogami and Y. Watabiki, Phys. Lett. B306 19 (1993).
23. A. Dasgupta and R. Loll, Nucl. Phys. B606 357 (2001).
24. J. Ambjørn, J. Jurkiewicz and R. Loll, Phys. Rev. D64 044011 (2001); Nucl. Phys. B610 347 (2001).
25. V. Kazakov and P. Zinn-Justin, Nucl. Phys. B546 647 (1999).
26. J. Ambjørn, J. Jurkiewicz, R. Loll and G. Vernizzi, JHEP 0109 022 (2001).